Displacement–length scaling in three dimensions: the importance of aspect ratio and application to deformation bands

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Abstract

Deformation bands confined to a 9-m-thick layer of the Entrada Sandstone in Utah accumulate less displacement per unit length than fractures that are not stratigraphically confined. This difference in displacement–length ($D$–$L$) scaling is related to the increasing length-to-height (aspect) ratio of the bands. Here we derive new expressions for displacement–length scaling and fracture strain for three-dimensional (3-D) elliptical fractures. The maximum (relative) displacement $D_{\text{max}}$ on an elliptical fracture surface depends on the fracture geometry (both length $L$ and height $H$), the end-zone length (through driving stress and the rock’s yield strength), and the properties of the surrounding rock (modulus, Poisson’s ratio). A given elliptical fracture will show different values of $D_{\text{max}}/L$ in horizontal and vertical sections due to differences in fracture dimension (length vs. height) and end-zone length. A population of elliptical fractures can accommodate less displacement or strain if fracture aspect ratios increase with $L$ than a population of fractures having constant aspect ratios. These relationships reveal how 3-D fracture geometries systematically influence the population statistics. The magnitude of horizontal (extensional) fracture strain accommodated by the population of deformation bands predicted by the analysis is consistent with that obtained independently from traverse measurements on the outcrop (0.12%). The 3-D fracture geometry can contribute at least an order of magnitude in displacement deficit (or excess) relative to tall 2-D fractures and comparable scatter on maximum displacement vs. length ($D$–$L$) diagrams. In general, fractures confined to stratigraphic layers grow nonproportionally ($L/H \neq$ constant for $L >$ the layer thickness), leading to reduced capacity to accommodate displacement and a shallower slope on the $D$–$L$ diagram. Similarly, fractures that grow by segment linkage (preferentially down-dip or along-strike) scale as nonproportional 3-D fractures. A unit slope on $D$–$L$ diagrams implies proportional growth ($L/H = $ constant). Faults with slip surfaces and other fractures with nonpreferred growth directions can produce unit slopes, so that particular trajectories on $D$–$L$ diagrams can reveal physical controls on fracture growth, such as stratigraphic confinement or segment interaction and linkage. © 2002 Published by Elsevier Science Ltd.

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1. Introduction and background

Displacement-length ($D$–$L$) scaling of fractures is understood by using two-dimensional (2-D) fracture mechanics models (e.g. Cowie and Scholz, 1992b) that assume plane conditions (e.g. Irwin, 1962; Heald et al., 1972; Palmer and Rice, 1972; Rudnicki, 1980; Kannel and Popelar, 1985, pp. 69, 153; Bürgmann et al., 1994). In rock mechanics, a 2-D analysis idealizes a volume with fractures as a rectangular plate in an $xy$ (Cartesian) coordinate system, with the vertical (z, ‘thickness’) dimension included only implicitly (Timoshenko and Goodier, 1970, pp. 15–17; Tada et al., 2000, pp. 14–15). The thickness of a ‘thin’ (plane stress) 2-D plate is negligible compared with the in-plane ($xy$) dimensions, so that in-plane stresses ($\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$) can produce warping and strain of the plate in the out-of-plane ($z$) direction. In contrast, the thickness of a ‘thick’ (plane strain) plate is sufficiently large to prevent strain in the out-of-plane ($z$) direction in response to the in-plane stresses. A fracture in a 2-D analysis is oriented perpendicular to the in-plane directions so that variations in stress or displacement in the $z$-direction are not introduced (e.g. Paris and Sih, 1965; Anderson, 1995, pp. 23, 36, 56). Looking down the thickness ($z$) direction onto the surface of the plate, the planar fracture is represented as a line of length $L$ within the $xy$ coordinate system. Because the vertical (third, $z$) dimension is not considered explicitly in a 2-D stress analysis, neither is the fracture’s height (its dimension in the $z$-direction, $H$). In that 2-D analysis, the length and displacement are associated (Fig. 1) without directly

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considering the influence of fracture height, leading to the standard $D$--$L$ scaling relations that only incorporate displacement and length (e.g. Muraoka and Kamata, 1983; Cowie and Scholz, 1992a,b; Clark and Cox, 1996; Fossen and Hesthammer, 1997; Renshaw, 1997; Gudmundsson, 2000; Bonnet et al., 2001).

In contrast, a 3-D stress analysis of a fractured rock mass explicitly considers dimensions, stresses, and displacements in all three $(xyz)$ directions, including the fracture’s height and shape. As noted by Segall and Pollard (1980), Scholz (1982), Olson (1993), Nicol et al. (1996), Willemsen (1997), Martel and Boger (1998), Kattenhorn et al. (2000), Schultz (2000), and others, fracture heights can be much smaller than (or comparable with) their lengths, leading to different $D$--$L$ scaling relations (Willemsen et al., 1996). Indeed, several studies have shown that the shorter dimension controls the displacement magnitude and associated fracture interaction distance (Segall and Pollard, 1980; Pollard and Segall, 1987; Olson, 1993; Pollard et al., 1993; Willemsen, 1997; Cowie, 1998; Crider and Pollard, 1998; Gudmundsson, 2000).

A 3-D fracture is defined in this paper as an elliptical surface with specified values of length, height, dip, and maximum relative displacement in a 3-D $xyz$ coordinate system (e.g. Irwin, 1962; Kassir and Sih, 1966; Willemsen, 1997; see Tada et al. (2000, pp. 14–15) for terminology particular to the small region near the fracture tip). Physically, the maximum relative displacement ($D_{\text{max}}$) on a fracture depends explicitly on both length $L$ and height $H$ (e.g. Irwin, 1962; Kassir and Sih, 1966; Chell, 1977; Willemsen et al., 1996; Willemsen, 1997; Martel and Boger, 1998), implying that the $D$--$L$ scaling relations may differ for a 3-D fracture from those that ignore fracture height (the 2-D plane approach discussed above). An explicit consideration of the elliptical shapes of natural (3-D) fracture surfaces can be critically important for understanding the stresses and displacements in the vicinity of the fracture (e.g. Olson, 1993; Willemsen, 1997; Crider and Pollard, 1998; Kattenhorn et al., 2000), and by implication, the balance between displacement accumulation and propagation of the fracture’s tipline (Rubin, 1992; Willemsen and Pollard, 2000). We demonstrate in this paper how the $D$--$L$ scaling relations depend on both dimensions (length and height) of the fracture plane.

Deformation bands from the Jurassic Entrada Sandstone of southeastern Utah accumulate less displacement per unit length than do well-developed faults (Fossen and Hesthammer, 1997, 1998). These cataclastic deformation bands (Aydin, 1978b; Aydin and Johnson, 1978; Davis, 1999; Fossen, 2000) are confined to a 9-m-thick (Aydin, 1978a) porous sandstone layer within the Entrada Sandstone (Fig. 2) and also lack slip surfaces (Fossen and Hesthammer, 1998); deformation bands that accommodate larger, discontinuous displacements (of more than several meters) along corrugated slickensided slip surfaces transect the Entrada Sandstone (Aydin, 1978b; Fig. 2c). The deformation bands that lack slip surfaces scale according to $D_{\text{max}} = \gamma L^{0.5}$ (Fossen and Hesthammer, 1997), whereas other fault sets scale with an exponent closer to 1.0 (e.g. Cowie and Scholz, 1992a; Clark and Cox, 1996). Fossen and Hesthammer (1997, 1998) hypothesized that the smaller exponent ($-0.5$) for the deformation bands could be attributed to a lesser degree of strain localization, with the exponent increasing to 1.0 as they evolve into surfaces of discontinuous displacement.

In this paper we derive displacement--length scaling relations for elliptical fractures and apply them to stratigraphically confined deformation bands. We show that the maximum relative displacement ($D_{\text{max}}$) along a fracture is a continuous, explicit function of both its length and height, similar to the dependence of stress magnitude near an elliptical cavity given by the well-known Inglis/Kolosov relation (e.g. Timoshenko and Goodier, 1970, pp. 191–194; Rekash, 1979, pp. 134–138). We then explore the characteristics of this 3-D scaling relation on standard displacement--length diagrams and demonstrate that lines of constant 1.0 slope (with the associated values of fracture shear strain $\gamma$) are a special case of 3-D scaling in which aspect ratios remain constant throughout the population. The results reveal that the shallower (0.5) slope obtained from deformation bands reported by Fossen and Hesthammer (1997) is related to their 3-D geometry, and we discuss the implications of 3-D $D$--$L$ scaling for fracture growth within stratigraphic layers and by segment linkage.
2. The 3-D deformation band data set

Deformation bands in the Entrada Sandstone are narrow tabular zones of localized compactional and shearing displacement in an otherwise homogeneous stratigraphic layer (Aydin, 1978b; Fossen and Hesthammer, 1997; Davis, 1999). Recent work demonstrates that displacement distributions along deformation bands vary systematically with position (Antonellini and Aydin, 1995; Fossen and Hesthammer, 1997, 1998; Wibberley et al., 2000), similar
Fig. 3. (a) Deformation bands exposed as narrow ridges in subhorizontal outcrop of the porous yellow-gray sandy layer \( e_1 \) in Entrada Sandstone. (b) Cross-sectional view of layer \( e_1 \) showing 9 m thickness, \( T \), subvertical bands, and reddish silty layers (\( e_2 \) and \( e_5 \) in Fig. 2c) above and below. Note graduate students on skyline for scale. (c) View along lower contact of layer \( e_1 \) with Carmel Formation (layer \( c_3 \)) showing deformation bands (subvertical surfaces between blocks) terminating against underlying layer \( c_5 \). (d) Detail of lower contact of layer \( e_1 \) showing deformation bands decreasing in width toward \( c_5 \); note pencil for scale. (e) Normal fault in Entrada Sandstone ('Fault 1' in Fig. 2b and c) with corrugated slip surface and 6 m of dip-slip offset. Note cutoff of layers \( e_1 \) and \( e_5 \) (upper right of image) down on the left of the slip surface that truncates the network of deformation bands in its hanging wall, the high frequency of deformation bands visible in layer \( e_1 \), and the absence of deformation bands in the underlying layers. Geologist (circled) on hanging wall for scale.
to those of other fracture types (e.g. Muraoka and Kamata, 1983; Walsh and Watterson, 1988; Bürgmann et al., 1994; Gupta and Scholz, 2000). Displacement maxima occur far from band terminations where near-tip displacement gradients and band geometries (in outcrop) are consistent with mechanical interaction and band linkage (Cruikshank et al., 1991; Antonellini and Aydin, 1995; Fossen and Hesthammer, 1997; Willemse, 1997). Propagation of deformation bands appears related to stress changes induced near their terminations (e.g. Du and Aydin, 1993; Antonellini and Pollard, 1995) that, in turn, scale with the magnitude of displacements accommodated across them. Although the spacing between cataclastic deformation bands may be related to strain hardening within the bands and the remote strain distribution and magnitudes imposed on them (e.g. Antonellini et al., 1994), their lengths are related to their displacement distributions (Fossen and Hesthammer, 1997, 1998).

Nucleation and growth of deformation bands in sandstones are sensitive initially to porosity, grain size and sorting, presence of clay minerals, confining pressure, and differential stress and, with ongoing deformation, to the changes in these characteristics (e.g. Aydin and Johnson, 1983; Antonellini et al., 1994; Menéndez et al., 1996; Mair et al., 2000; Olsson, 2000; Bésuelle, 2001; Issen and Rudnicki, 2001). Observations of deformation bands in the field demonstrate that they can be confined to particular stratigraphic layers, for which band localization is facilitated, with growth across stratigraphic contacts inhibited where they encounter layers having unfavorable conditions (e.g. Fossen, 2000). Confinement of deformation bands to particular layers (e.g. Fig. 2a) is similar to joints, stylolites, and other types of localized strain that are also highly dependent on local stress state and lithologic properties for their development. As stratabound fractures lengthen, their aspect ratios increase, whereas their heights remain fixed by the stratigraphic thickness.

The deformation bands considered in this paper are confined to a well sorted and highly porous (i.e. ~20%) eolian yellow-gray layer within the Entrada Sandstone (Figs. 2 and 3a). The layer ('e1' in Fig. 2c) is 9 m thick (Aydin, 1978a; and verified by us in the field in May 2001) and is bounded above and below by red siltly sandstone layers that lack well-developed deformation bands ('e2' and 'e3' in Figs. 2c and 3b). Close examination of the contact regions reveals that the deformation bands within the yellow-gray layer consistently terminate at or just beyond the contacts with the red siltly sandstone layers (Fig. 3c and d). The deformation bands span a range of lengths (0.06 < L < 104 m; Fossen and Hesthammer, 1997) but are confined to the yellow-gray layer (e1), requiring that their heights be no more than the layer thickness, or H ≤ 9 m. As a result, the three-dimensional shapes (length vs. height) of the bands scale with their lengths, with bands longer than 9 m becoming progressively less tall in relation to their lengths (Fig. 4c). The aspect (length/height) ratios of the deformation bands in the study area are largest for the longest bands (L/H = 104/9 m = 11.6), giving the bands a range of three-dimensional elliptical shapes. Faulted deformation bands that accommodate much larger displacements than those investigated in this paper (i.e. meters) are characterized by discontinuous displacement across their corrugated and slickensided surfaces (e.g. Aydin and Johnson, 1978; Fossen and Hesthammer, 1997; Shipton and Cowie, 2001). These faulted bands cut across

![Diagram](image-url)

**Fig. 4.** (a) Geometry of dipping fracture planes within a layer. 3-D planar surfaces have measurable horizontal (trace) lengths and heights (measured in their planes); 2-D surfaces only provide measurements of their lengths. (b) Geometric parameters for an elliptical fracture in 3-D, with trace length L being twice the semi-horizontal axis (L = 2a) and height H being twice the semi-vertical axis (H = 2b) of the dipping plane. A fracture having its height much greater than its length (2b ≫ 2a) is a ‘tall’ or ‘tunnel’ fracture that penetrates deeply into the rock mass relative to its length. A circular, equidimensional fracture (2a = 2b) is referred to as a ‘penny’ fracture. A ‘long’ fracture has 2a ≫ 2b. For a 2-D plane fracture, height (2b) is indefinitely large. (c) Diagram showing how fractures larger than the layer thickness T can grow only in length at constant height, leading to nonproportional growth and 3-D scaling.
stratigraphic contacts in the study area (e.g. Krantz, 1988; Fossen, 2000, p. 32; Figs. 2c and 3e).

3. Maximum displacement vs. fracture aspect ratio

We idealize a fracture as a degenerate (flat) ellipse having length \( L = 2a \), height \( H = 2b \), and a (much smaller) magnitude of relative displacement, with the maximum value of the relative-displacement distribution given by \( D_{\text{max}} \) (Fig. 4). A deformation band is modeled as an elliptical fracture. Length is defined in this paper as the horizontal trace (or map) length (Fig. 4a), whereas height is taken to be the vertical (i.e. down-dip) dimension of the fracture surface in its plane (cf. Wattersen, 1986; Walsh and Wattersen, 1988), following standard practice (e.g. Cowie and Scholz, 1992a,b; Dawers et al., 1993; Cartwright et al., 1995; Willemsen et al., 1996; Gudmundsson, 2000). We apply results from elasticity theory and post-yield Elastic–Plastic Fracture Mechanics (EPFM) to construct an equation that explicitly relates \( D_{\text{max}} \), length, and height, and also allow the length of the end zone(s) to vary as a free parameter, rather than being prescribed up-front as a constant (e.g. Cowie and Scholz, 1992b; Martel and Boger, 1998).

3.1. Elliptical (elastic) fracture shape

The stresses and displacements associated with an elliptical fracture in an infinite homogeneous elastic material are well known from Linear Elastic Fracture Mechanics (LEFM; e.g. Green and Sneddon, 1950; Irwin, 1962; Kassir and Sih, 1966; Xue and Qu, 1999; Zhu et al., 2001). The maximum (relative) displacement \( D_{\text{max}} \) on the LEFM fracture surface is given by:

\[
D_{\text{max}} = \frac{2(1 - \nu)}{G} \sigma_d \left( \frac{b}{E(a,b)} \right)
\]

in which \( \sigma_d \) is the driving stress acting on the fracture (see Table 1), \( \nu \) is Poisson’s ratio, \( G \) is the shear modulus (both of the host rock), and \( a \) and \( b \) are the semi-major and semi-minor axes of the fracture (Fig. 4b), respectively. Driving stress is defined as the net traction (thought of as acting on the fracture) that leads to relative displacement of fracture

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Driving and resisting stresses for geologic fractures</th>
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<td>Term</td>
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* Remote stresses resolved trigonometrically onto potential fracture plane; values applicable before fracture deforms and accommodates relative displacements.

\( \nu \) DB: deformation band.

b.c.: boundary condition.
walls and concomitant amplification of stresses along the fracture’s tip (in 2-D) or elliptical trilene (in 3-D). For joints and dikes (Table 1), $\sigma_d$ is the resolved remote normal stress component (usually compressive) minus the internal pore-fluid pressure within the fracture (Pollard and Aydin, 1988; Rubin, 1993). For faults, $\sigma_d$ is the maximum (static) frictional strength minus the residual frictional strength of the surface (Palmer and Rice, 1972; Rudnicki, 1980; Cooke, 1997). For deformation bands, the driving stress is related to the remote compressive and shear stress components resolved onto the band and the resistance of the band to compaction (Das, 1983, p. 36), cataclasis, and localized shearing $\sigma_f$ (Table 1).

$E(a,b)$ in Eq. (1) is the complete elliptic integral of the second kind (e.g. Irwin, 1962; Kanninen and Popelar, 1985, p. 153; Lawn, 1993, p. 33), which is given by:

$$E(a, b) = \int_0^{\pi/2} \sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)^2} \sin \varphi \, d\varphi$$

where $\varphi$ is amplitude of the elliptic integral (Boas, 1966, pp. 410–413). Particular values of Eq. (2) can be obtained by specifying $a$ and $b$ and using standard mathematical tables to evaluate the integral (e.g. Beyer, 1987). For a circular, ‘penny-shaped’ fracture ($a = b$), $E(a,b) = \pi/2$, whereas a long, 2-D fracture ($a \gg b$) yields $E(a,b) = 1$ (see also Gudmundsson, 2000). This ‘shape factor’ (Eq. (2)) reduces both $D_{\text{max}}$ and the associated stress intensity factor relative to those for a tall 2-D fracture (e.g. Lawn, 1993, pp. 31–33). The maximum displacement on an elliptical fracture (Eq. (1)) thus depends explicitly on both $a$ and $b$ (Irwin, 1962; Willems et al., 1996; Gudmundsson, 2000). Eq. (2) can be approximated for fracture mechanics problems by a flaw shape parameter (Anderson, 1995, pp. 115–116) $\Omega$:

$$E(a, b) = \Omega \equiv 1 + 1.464 \left(\frac{a}{b}\right)^{1.65}$$

This approximation is within 5% of the numerical solution to Eq. (2) obtained from tabulated values listed in Beyer (1987), as demonstrated in Fig. 5. $\Omega$ can be substituted into Eq. (1) for $E(a,b)$, permitting calculation of $D_{\text{max}}$ for arbitrary fracture lengths and aspect ratios.

For an elliptical fracture, the half-length parameter $b$ in Eq. (1) must be modified to vary from $a$ to $b$ (Irwin, 1962; Kassir and Sih, 1966). Using the factor:

$$\sqrt{\sin^2 \theta + \left(\frac{a}{b}\right)^2 \cos^2 \theta}$$

in which $\theta$ is the angle to a point on its trilene (Irwin, 1962; Kassir and Sih, 1966; Fig. 4b), the radius of the ellipse $r$ (defined as the distance from the ellipse’s center to a point on its trilene) is given by:

$$r = b \sqrt{\sin^2 \theta + \left(\frac{a}{b}\right)^2 \cos^2 \theta}$$

The equation for maximum (relative) displacement on an elliptical fracture in an elastic medium, subjected to a uniform driving stress (and assuming negligibly small end zones in LEFM) is obtained by combining Eqs. (1), (3), and (4) to produce:

$$D_{\text{max}} = \left[\frac{2(1 - \nu)}{G} \sigma_d \right] \sqrt{\frac{b}{1 + 1.464 \left(\frac{a}{b}\right)^{1.65}}} \left(\frac{b}{a}\right)^2 \cos^2 \theta$$

Eq. (6) reduces to 2-D cases for the limiting values of $r = a$ and $r = b$. For a slice along the fracture’s horizontal axis (the 2D direction), $\theta = 0^\circ$ and Eq. (5) yields a half-length of $a$. For a vertical slice (along the 2b axis), $\theta = 90^\circ$ and Eq. (5) yields a half-length of $b$. Similarly, for a tall 2-D, plane-strain fracture (with $b \gg a$), the denominator (Eq. (3)) approaches 1.0; the increase in Eq. (3) with larger values of aspect ratio $a/b$ (Fig. 5) controls the well-known reduction in $D_{\text{max}}$ with elliptical geometry (specifically, with $b/a$; Irwin, 1962; Willems et al., 1996). Eq. (6) implies that the $D-L$ scaling of 3-D elliptical fractures depends on both length $2a$ and height $2b$. 
3.2. End zone and effective driving stress

Consideration of end zones is important in 3-D $D-L$ scaling for several reasons:

1. Most geologic fractures do not reflect conditions of small-scale yielding and LEFM (Cowie and Scholz, 1992b; Rubin, 1993) due to a combination of finite (small) rock strength at the tipline and relatively large ratios of driving stress to yield strength (e.g. Rubin, 1993). Measurements of the near-tip displacement profiles for joints and dikes (Rubin, 1993; Khazan and Fialko, 1995), faults (Dawers et al., 1993; Cartwright and Mansfield, 1998; Cowie and Shipton, 1998; Moore and Schultz, 1999), and deformation bands (Fossen and Hesthammer, 1997) reveal large departures from the ideal steep near-tip displacement gradients required by LEFM, such as tapered or linear displacement distributions, whose maximum values ($D_{\text{max}}$) are also smaller than those of an equivalent LEFM fracture.

2. End zones provide a physical basis for the linear displacement–length scaling of geologic fractures (Cowie and Scholz, 1992b). By replacing fracture toughness (having units of MPa m$^{1/2}$) by yield strength (having units of MPa), a linear (proportional) relationship between the maximum displacement and fault length (e.g. Clark and Cox, 1996) is achieved. $D_{\text{max}}$ would scale with $L^{1/2}$, instead of $L$, for an LEFM fracture.

3. The effective driving stress $\sigma_d$ associated with $D_{\text{max}}$ on an elastic–plastic fracture depends on the length of the end zone, which in turn depends on both the ratio of driving stress $\sigma_d$ (over the fracture’s fully yielded part) to yield strength $\sigma_y$ (over the end zone; Bilby et al., 1963; Heald et al., 1972; Chell, 1977; Li and Liang, 1986) and the fracture’s aspect ratio (Chell, 1977). Although some workers specify particular values of end-zone length (e.g. Cowie and Scholz, 1992b; Martel and Boger, 1998), others treat it as a free parameter given uncertainties in actual values of driving stress $\sigma_d$ or yield strength $\sigma_y$ (e.g. Heald et al., 1972; Rudnicki, 1980; Li and Liang, 1986; Rubin, 1993; Bürgmann et al., 1994; Cooke, 1997; Martel, 1997). As a result, the effective driving stress $\sigma_d$ on an elastic–plastic fracture depends explicitly on both $a/b$ and $\sigma_y/\sigma_d$.

In this section we develop expressions for end-zone length around the tipline of an elliptical fracture as a function of aspect ratio $a/b$ and end-zone strength $\sigma_y/\sigma_d$.

We build on the approach pioneered for 2-D plane problems by Dugdale (1960) and Barenblatt (1962) for mode-I cracks, Palmer and Rice (1972) for mode-II slip surfaces in overconsolidated soil, and Cowie and Scholz (1992b) for faults in rock. Although the near-tip deformation mechanisms invoked by Dugdale (1960) and Barenblatt (1962) were different (cohesive vs. plastic), the form of their results is applicable to fractures having non-LEFM ‘end zones’ in which the specific yielding mechanism may depend on the fracture displacement mode (e.g. Cowie and Scholz, 1992b; Rubin, 1993). Heald et al. (1972), Chell (1977), Turner (1979), Li and Liang (1986) and Ingraffea (1987) discuss applications of fractures with sizable (non-LEFM) end zones in engineering, whereas Rudnicki (1980), Li (1987), Martel and Pollard (1989), Cowie and Scholz (1992b), Rubin (1993), Bürgmann et al. (1994), Cooke (1997), Martel (1997), Cowie and Shipton (1998) and Willenme and Pollard (1998) apply the approach to 2-D fractures in rock. End zones around equidimensional, ‘penny-shaped’ fractures are calculated and discussed by Chell (1977) and Martel and Boger (1998). In all cases, fracture–surface displacements are modulated by end zones that are sufficiently large that the limiting LEFM solution is not appropriate. Mechanically, the end zones attain lengths that exceed the Irwin plastic-zone dimensions when the requirement for small-scale yielding conditions is relaxed (e.g. Kanninen and Popelar, 1985, pp. 282–283). This also modifies the shape of the relative-displacement distribution, from elliptical to bell-shaped or linear (e.g. Cowie and Scholz, 1992b; Bürgmann et al., 1994; Cooke, 1997).

We define the driving stress $\sigma_d$ on a deformation band as $(\sigma_p - \sigma_i)$ and the yield strength $\sigma_y$ of the end zone as $(\sigma_p - \sigma_i)$ (see Table 1 and Fig. 6). We associate the end zone bounding a deformation band with the area ahead of the band within which the porosity locally increases and

![Fig. 6. Distribution of stress (resolved on the fracture plane) near the tip of an elastic–plastic fracture. Amplified stress due to fracture-wall displacements (dashed curves) decays with distance from fracture tip to $\sigma_i$ in the surrounding unfractured rock. Length $s$ of end zone is defined by a constant value of yield (peak) stress $\sigma_y$. Driving stress $\sigma_d$ is the difference between resolved remote stress $\sigma_p$ and internal boundary value $\sigma_i$ (e.g. pore-fluid pressure or residual frictional strength). Yield strength $\sigma_y$ is the difference between $\sigma_p$ and internal boundary value $\sigma_i$.](image-url)
grain fracturing (for cataclastic bands) begins (e.g., Aydin, 1978b; Antonellini et al., 1994) in response to amplified stresses near the band’s tipline (Fig. 6).

Sizable end zones are associated with a reduction in the value of $D_{\text{max}}$ on the fracture (e.g., Bürgmann et al., 1994) relative to the LEFM case that incorporates indefinitely strong (and negligibly small) end zones with a fracture of equivalent dimension. As emphasized by Rubin (1993), however, the length of an elastic–plastic fracture can be defined as either including or excluding the end zone (Fig. 7, central panel). The former case corresponds to the effective fracture length $L = 2(c + s) = 2a$ in LEFM, in which $s$ is the length of the (small) plastic zone at the fracture tip and $a = (c + s)$ (see Fig. 7). Cowie and Scholz (1992b) adopt $L$ as the effective fault length for $D$–$L$

scaling, following Rudnicki (1980) and Palmer and Rice (1972). On the other hand, the latter case that defines the fracture length $L^*$ by the ‘unyielded’ central portion $(c)$, excluding the end zone $(s)$, is commonly used in engineering fracture mechanics (e.g., Bilby et al., 1963; Heald et al., 1972; Chell, 1977), with $L^* = 2c$. In LEFM, $c \rightarrow a$, leading to little ambiguity in locating the fracture tip precisely, but this may not be the case for geologic fractures (e.g., Rubin, 1993). In this paper we adopt the total, effective length $L = 2(c + s)$, where $s$ is the explicit, non-negligible length of the end zone whose length may exceed the (negligibly small) limiting value $s$ given in LEFM.

In order to balance the singular near-tip elastic stresses generated near an ideal fracture (and to eliminate the LEFM singularity; Goodier, 1968), opposing stresses are defined (using a Dugdale-type ‘cohesive’ end zone conceptualization) to act across a narrow zone, coplanar with the fracture. The length $s$ of the end zone associated with a 2-D fracture, written as the ratio of nonyielded $(c)$ to effective $(a)$ fracture lengths (Bilby et al., 1963; Heald et al., 1972; Chell, 1977), is:

$$\frac{c}{a} = \cos \left( \frac{\pi \sigma_d}{2 \sigma_y} \right)$$  \hspace{1cm} (7a)

or explicitly,

$$\frac{s}{a} = 1 - \cos \left( \frac{\pi \sigma_d}{2 \sigma_y} \right)$$  \hspace{1cm} (7b)

in which $\sigma_d$ is the driving stress and $\sigma_y$ is the (constant) yield strength of the surrounding rock ahead of the fracture tip (and in the end zone). The cosine function in Eqs. (7a) and (7b) relates the quantity $\sigma_d/\sigma_y$ to the dimensions for nonyielded fracture $c$, end zone $s$, and total effective (elastic–plastic) fracture half-length $a$ (Fig. 7, upper panel). For $\phi = [(\pi/2)(\sigma_d/\sigma_y)] = 0^\circ$, $\sigma_y \rightarrow \infty$, $s \rightarrow 0$, and LEFM conditions apply (Fig. 7, lower panel, right-hand side). Smaller values of $\sigma_y$ are associated with greater end-zone lengths and larger values of $\phi$. These expressions are considered appropriate for $(\sigma_d/\sigma_y) > 1.1$ (Chell, 1977; Kanninen and Popelar, 1985, p. 283), implying maxima for $s/a < 0.86$ and $\phi < 82^\circ$ (dark shaded region in Fig. 7, lower panel). For the particular case of $s = 0.2a$ adopted by Cowie and Scholz (1992b) for faults (light shading in Fig. 7), $\phi = 37^\circ$.

End-zone length (Eq. (7b)) scales with driving stress and inversely with yield strength. As demonstrated in Fig. 8a, as the driving stress becomes small relative to the yield strength (i.e., $(\sigma_d/\sigma_y) > 10$), $c/a \rightarrow 1$ and $s/a \rightarrow 0$, approximating LEFM conditions (right-hand side of Fig. 8). However, as the driving stress becomes larger relative to the yield strength of the rock in the end zone, $c/a$ decreases and the end zone grows relative to the total effective fracture half-length $a$. In this case, the effective driving stress acting

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*Fig. 7. Upper panel relates the quantity $\sigma_d/\sigma_y$ to dimensions for yielded 2-D fracture half-length $c$, end zone length $s$, and total effective (elastic–plastic) fracture half-length $a$ using the parametric angle $\phi = [(\pi/2)(\sigma_d/\sigma_y)]$ (after Kassir and Sih, 1966; Tada et al., 2000, p. 384). Lower panel shows dependence of $s/a$ and $c/a$ on $\sigma_d/\sigma_y$; $s/a \leq 0.2$ shaded; expressions valid for $s/a < 0.86$ and $\phi < 82^\circ$ (dark shaded region).*
over the elastic–plastic ('post-yield') fracture includes the competing contributions from the driving stress $\sigma_d$ that acts over the entire length $L = 2a$, promoting relative displacement of the fracture walls, and from the opposing stresses within the end zone (defined by $\sigma_{xz} = (\sigma_d - \sigma_y)$; Table 1). We define an effective driving stress $\sigma_d^*$ acting over the effective elastic–plastic fracture length $2a$, for 2-D fractures, as:

$$\sigma_d^* = \left[ \sigma_d \left( \frac{c}{a} \right) \right] + \left[ \sigma_{xz} \left( \frac{s}{a} \right) \right]$$  

(8a)

in which $a$ is a constant for the 2-D fracture. Using Eqs. (7a) and (7b) and rearranging, Eq. (8a) becomes:

$$\sigma_d^* = \sigma_d - \sigma_y \left[ 1 - \cos \left( \frac{\pi \sigma_d}{2 \sigma_y} \right) \right]$$  

(8b)

The effective driving stress given by Eqs. (8a) and (8b) is implicitly defined for unit half-length $a$ (see Fig. 7a). The effective driving stress for a 3-D elliptical fracture is given by:

$$\sigma_{de}^* = \sqrt{\frac{\sigma_d - \sigma_y \left[ 1 - \cos \left( \frac{\pi \sigma_d}{2 \sigma_y} \right) \right]}{\left( \cos^2 \theta + \left( \frac{b}{a} \right)^2 \sin^2 \theta \right) \left[ 1 + 1.464 \left( \frac{a}{b} \right)^{1.65} \right]}}$$  

(9)

The term $\sigma_d^*$ in Eqs. (8a) and (8b) corresponds to that used by Cowie and Scholz (1992a, see Table 1) and Scholz (1997) with, however, the explicit specification of an important variable coefficient. For the particular case discussed by Cowie and Scholz (1992b), $s = 0.2a$ and $\sigma_d/\sigma_y = 2.44$ (see Fig. 8), Eq. (8b) requires $\sigma_d^*/\sigma_d = 0.51$ (Fig. 8b). By explicitly evaluating the effective driving stress $\sigma_d^*$ over the total effective (elastic–plastic) fracture length $2a$ using Eqs. (8a) and (8b), the decrease in $D_{max}$ with increasing $s$ (for a given fracture length $2a$ and shown explicitly by Bürgmann et al. (1994)), is related to the reduction in effective driving stress $\sigma_d^*$ acting over the fracture relative to $\sigma_d$ (see Fig. 8b). The discrepancy between the effective ($\sigma_d^*$) and LEFM-based driving stress ($\sigma_d$) increases with progressively smaller (weaker) values of yield strength. For example, the discrepancy in driving stress is ~10% for fractures having $s = 1\%$ of $a$ (Fig. 7a; $s/a = 0.01$) and >50% for $s/a > 0.2$. This variable coefficient (the vertical scale in Fig. 8b) can be incorporated into the parameter $C$ in the Cowie and Scholz (1992a) term for fracture shear strain, $\gamma = C(\sigma_y - \sigma_d)/G$ (see also Scholz, 1997). By explicitly considering the relationship between effective driving stress and end-zone length (Eq. (8b)), the curves for the curves for 3-D $D-L$ scaling developed in this paper shift to smaller values than the approximate 2-D contours of $\gamma$ (diagonal dashed lines in Fig. 1) by the factor $\sigma_d^*/\sigma_d$.

The change in geometry from 2-D (tunnel) to 3-D (penny) fractures is associated with a decrease in end-zone length relative to 2-D fractures (Fig. 9a). As shown in Fig. 9b, assuming the same values for fracture length ($2a$), driving stress, and yield strength, the end zone adjacent to a penny crack is consistently smaller than that bounding a 2-D tunnel crack. The smaller end-zone length is due to the greater constraint on deformation imposed on the penny crack by its fully bounded circular periphery, compared with the tunnel crack that is bounded only along its vertical edges (Chell, 1977; Martel and Boger, 1998). The reduction of end-zone length with fracture geometry is associated physically with a corresponding decrease in the value of maximum displacement $D_{max}$ for the penny crack relative to the 2-D tunnel crack ($2/\pi$; e.g. Lawn, 1993, p. 32). Tall elliptical fractures ($b > a$) should have values of end-zone length intermediate between the tunnel and penny geometries (e.g. Fig. 9b).

By substituting Eq. (5) into Eq. (7b), the length of the end zone surrounding an elliptical fracture is given...
approximately by:

\[
s = \left[ 1 - \cos \left( \frac{\pi}{2} \frac{\sigma_d}{\sigma_y} \right) \right] \left( b \frac{\sin^2 \theta + \left( \frac{a}{b} \right)^2 \cos^2 \theta}{1 + 1.464 \left( \frac{a}{b} \right)^{1.65}} \right)
\] (10)

As shown in Fig. 9b, the predicted length is within ±30% of that for the penny fracture as given by Chell (1977) and Martel and Boger (1998). This difference is reasonable given the idealization of near-tip yielding within a volume as a planar zone (e.g. Bilby et al., 1963; Heald et al., 1972; Li and Liang, 1986; Engelder et al., 1993) and uncertainties of perhaps a factor of two for field estimates of the length of an equivalent planar end-zone (e.g. Cowie and Scholz, 1992b).

Because the magnitude of \( D_{\text{max}} \) is proportional to the level of driving stress acting on the fracture (e.g. Pollard and Segall, 1987; Eq. (1)), and for a given fracture size and yield strength, end-zone length increases with \( D_{\text{max}} \). Longer end zones are generated to absorb the greater magnitudes of near-tip stress associated with larger values of \( D_{\text{max}} \). Fractures of comparable length but with smaller values of \( D_{\text{max}} \) would require smaller end zones (e.g. Li and Liang, 1986).

4. Characteristics of the elliptical (3-D) elastic–plastic fracture

By combining the expressions for the maximum (relative) displacement on an elliptical fracture in an elastic medium (Eq. 6) with the effective driving stress for an elastic-plastic fracture (Eq. (9)), the 3-D displacement-length scaling relation becomes

\[
D_{\text{max}} = \left[ \frac{2(1 - \nu)}{G} \right] \times \frac{\sigma_d - \sigma_y}{\sigma_y} \left[ 1 - \cos \left( \frac{\pi}{2} \frac{\sigma_d}{\sigma_y} \right) \right] \left( b \frac{\sin^2 \theta + \left( \frac{a}{b} \right)^2 \cos^2 \theta}{1 + 1.464 \left( \frac{a}{b} \right)^{1.65}} \right)
\] (11)

Scaling of maximum displacement and length for 3-D elliptical fractures (Eq. (11)) depends on the properties of the surrounding rock (modulus and Poisson’s ratio), the end-zone length (through driving stress and yield strength), and the ellipse geometry (\( \theta, a \), and \( b \)). The form of Eq. (11) is similar to previous 2-D formulations (e.g. Cowie and Scholz, 1992b) with the displacement–length ratio

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Fig. 9. (a) Comparison of theoretical end-zone lengths from the literature. Dashed line shows \( sl/a = 0.2 \). (b) Fracture aspect ratio \( a/b \) reduces end-zone length at a given end-zone strength for 3-D (penny) fractures (\( a/b = 1 \); lower curve) relative to 2-D (tunnel) fractures (\( b \gg a \); upper curve). Filled squares show values calculated by Martel and Boger (1998); coincident curve for penny fracture is from Chell (1977). Filled dots and diamonds show values calculated using Eq. (10) for tunnel and penny geometries, respectively; error bars show ±30% of Chell’s (1977) and Martel and Boger’s (1998) values for the penny geometry. (c) Relative length of end zone for tunnel fracture, compared with penny fracture, as a function of yield strength. Horizontal shaded bars in (b) and (c) indicate range of predicted end-zone strengths for length \( s = 0.2a \). Vertical bars in (a), (b), and (c) show limits of validity for theoretical end-zone lengths (\( \sigma_y/\sigma_d < -1.1 \)).
results reported by Willemse et al. (1996), as shown in Fig. 10. Relative to an indefinitely tall 2-D fracture \((b \gg a)\), a penny fracture can accommodate only about 60\% \((2/\pi)\) of the displacement on the tall fracture (large dot on Fig. 10), whereas an indefinitely long 2-D fracture \((a \gg b)\) can accommodate less than one-tenth of the displacement on the tall fracture of the same size (e.g. for \(a/b > 8\) in Fig. 10). As a result, standard 2-D approaches to displacement–length scaling that neglect the effect of fracture height \((2b)\) may systematically overpredict the magnitude of \(D_{\text{max}}\) on a particular fracture by at least an order of magnitude. These errors will propagate through the population statistics if aspect ratio is not explicitly accounted for.

The end-zone length (Eq. (10)) varies as a function of position along the elliptical tinele, as shown in Fig. 11a for several values of end-zone strength \((\sigma_f/\sigma_0)\). The length at any point along the tinele depends additionally on that fracture’s aspect ratio. For example, \(s\) is maximum for long fractures \((a/b = 10\) along the vertical, semi-minor \((\theta = 90^\circ\) and \(r = b)\) axis and minimum along the horizontal direction (semi-major axis, \(\theta = 0^\circ\) and \(r = a)\). For tall fractures (Fig. 11b; inset to Fig. 11), \(s\) is maximum along the horizontal direction \((\theta = 0^\circ\) and \(r = a)\). Comparison with commonly cited values for \(s/a\) of \(-0.1\)–\(-0.2\) (shaded bar in Fig. 11a and b) suggests that knowledge of both the position along a fracture’s tinele \((\theta)\) and the fracture aspect ratio \((a/b)\) would be necessary to interpret the significance of end-zone length for a given value of strength.

The ratio of maximum displacement \(D_{\text{max}}\) to fracture length \(L\) depends on both the position along the tinele (Fig. 11c) and the fracture’s aspect ratio (Fig. 11d). \(D_{\text{max}}/L\) is maximized on fractures with stronger end zones and minimized on those with weak end zones, supporting previous suggestions (Cowie and Scholz, 1992b; Martel, 1997; Moore and Schultz, 1999; Schultz, 1999; Wiibberley et al., 2000) that the yield strength of an end zone can modulate the displacement–length scaling relations. Measurement of \(D_{\text{max}}\) and \(L\) in either map view or cross-section, for the same fracture, can produce substantial variation in \(D_{\text{max}}/L\) due to its sensitivity to the fracture’s aspect ratio.

4.2. The \(D–L\) diagram for 3-D fractures

The 3-D scaling relationship (Eq. (11)) can be plotted on the standard \(\log–\log D_{\text{max}}/L\) diagram (Fig. 12, for horizontal trace lengths, \(\theta = 0^\circ\)). Tall fractures \((L < 2b)\) define a positive 45\(^\circ\) (1:1) slope (left side of diagram, dashed lines) because \(D_{\text{max}}\) depends only on the length \(L\) (and not on the height \(H = 2b\), for given values of modulus and Poisson’s ratio). The change in slope on the \(D–L\) diagram, from 1:1 to shallower slopes, is related to the sensitivity of \(D–L\) scaling of 3-D elliptical fractures to the fracture height \(2b\) (e.g. Willemse et al., 1996; Gudmundsson, 2000; Fig. 10). As a fracture grows in length at constant height, its ability to accommodate displacements degrades with increasing

4.1. Variations in the main parameters

Willemse et al. (1996) compared the predicted values of \(D_{\text{max}}\) for elastic 2-D and 3-D cases that assume negligibly small end zones (i.e. LEFM conditions). They showed that \(D_{\text{max}}\) decreases systematically with increasing fracture aspect ratio \((a/b)\) but they did not develop the implications for displacement–length scaling in further detail. The particular values treated by Gudmundsson (2000) for tall \((b \gg a)\), circular \((a = b)\), and long \((a \gg b)\) fractures also match those used by Willemse et al. The scaling relationship for elliptical fractures defined in this paper (Eq. (11) with negligibly small end zones) compares favorably with the proportional to the yield strength of the rock ahead of the fracture tinele and inversely proportional to modulus \((G)\).

Eq. (11) extends the standard two-dimensional displacement–length scaling relation, for which fracture height \(2b\) is not considered (because \(2b \gg 2a\)) to 3-D elliptical fracture geometries. \(D_{\text{max}}\) is an explicit function of fracture length and fracture height in Eq. (11). Because \(a \neq b\) in general for natural fractures (e.g. Nicol et al., 1996; Gudmundsson, 2000), displacement–length scaling relations will necessarily depend on which length scale is chosen: the horizontal length or the vertical height. The relationship (Eq. (11)) assumes that the yield strength does not vary significantly along the fracture periphery (Chell, 1977), which appears reasonable for small fractures but perhaps not for fractures large enough to transect differing rheologies or for strongly interacting fractures (e.g. Gupta and Scholz, 2000).
Fig. 11. Diagrams showing how end zones and maximum displacement/length ratio vary systematically with position along an elliptical fracture trace and with fracture aspect ratio. Upper row displays variation in end-zone length (Eq. (10)) with (a) end-zone strength, and (b) fracture aspect ratio, both as a function of position along the trace of elliptical fracture. Shaded band in (a) and (b) shows typical range of end-zone lengths from the literature ($0.1 < \sigma_s / \sigma_d < 0.2$). Lower row displays variation in maximum displacement/length ratio with (c) end-zone strength, and (d) fracture aspect ratio, as a function of position along the trace. Shaded band in (c) and (d) shows typical range of displacement/length ratios for faults (e.g. Fig. 1). All calculations assume a yield strength of 10 MPa ($\sigma_y / \sigma_d = 10$), Poisson's ratio of 0.25, and shear modulus of 0.25 GPa.

aspect ratio, leading to slopes shallower than 1:1 on the $D-L$ diagram (right-hand side of Fig. 12).

A family of $D-L$ curves (calculated by using Eq. (11)) describes how maximum displacements on 3-D fractures of various heights scale with their map lengths (Fig. 12). The point on each curve that corresponds to an aspect ratio $a/b = 1$ (equidimensional or 'penny-shaped' fracture geometry) defines the transition from a normal (1:1) and constant slope to a shallower, nonlinear slope. Interestingly, connecting the points on different curves for the same value of aspect ratio (dots in Fig. 12b) produces a straight line, with a constant slope of 1.0. Other parallel lines can be constructed for other values of fracture aspect ratio (e.g. line for $a/b = 10$ in Fig. 12b). These lines defined by constant values of fracture aspect ratio describe fractures in a population that grow in a self-similar, proportional manner; i.e. at a constant aspect ratio. In 2-D $D_{max}/L$ scaling, these lines are related to the constant of proportionality $\gamma$.
between $D_{\text{max}}$ and $L$ (e.g., Cowie and Scholz, 1992b). In 2-D cases, the aspect ratio implicitly remains constant during fracture growth or for all members of the population because only the fracture length (and not its height) is considered. As Fig. 12 shows, the linear relationship in 2-D between $D_{\text{max}}$ and $L$ is a special case of 3-D scaling that occurs for geometrically self-similar fractures (i.e., constant aspect ratio, proportional growth).

The curves shown in Fig. 13 demonstrate how the maximum displacement for 3-D elliptical fractures varies (with $L$ and $a$ for given value of $b$) for a range of end zone yield strengths $\sigma_y$. $D_{\text{max}}$ shifts downward on the diagram (to smaller maximum displacements) for weaker end zones (at given values of $L$ and $2b$; labeled curves in Fig. 13), in accord with previous investigations of 2-D scaling (e.g., Cowie and Scholz, 1992b). For a given fracture height $2b$, fractures of length $L = 2a$ can build up greater values of maximum displacement as end-zone strength increases. However, the horizontal position of the inflection point in the curve ($a = b$) is unchanged. For example, very strong end zones ($\sigma_y/\sigma_d = 50$), approximating LEFM conditions and small-scale yielding ($a/a \rightarrow 0$), are associated with values of $D_{\text{max}} \sim 4$ times greater than comparable fractures that are bounded by weak end zones ($\sigma_y/\sigma_d = 1.5$). Thus, fractures in a strong rock, like granite, could accommodate larger displacements than those having identical dimensions and driving stresses in a soft rock like a sandstone (given comparable values of modulus and Poisson’s ratio for both rock types). For an actual fracture of given length and height, $D_{\text{max}}$ will reflect a balance between end zone strength, driving stress, and rock properties (e.g., Eq. (11)).

Lines of constant $D_{\text{max}}/L$ ratio obtained from 2-D $D-L$ scaling relations (Cowie and Scholz, 1992b; Clark and Cox, 1996) correspond to contours of fracture shear strain, $\gamma = C(\sigma_y - \sigma_d)/G$ (Cowie and Scholz, 1992b; Scholz, 1997). A fracture population that follows $\gamma = 0.01$ would
be characterized by values of maximum displacement that scale linearly with fracture length, with \( \gamma = 0.01 \) corresponding to a fracture shear strain (Jamison, 1989; Wojtal, 1989) of 1%.

The fracture shear strain that is accommodated in an elliptical fracture’s plane is given using the 3-D scaling relations by \( D_{\text{max}}/L = \alpha \). Specifically:

\[
D_{\text{max}} = \alpha L
\]

in which

\[
\alpha = \left[ \frac{(1 - \nu)}{G} \right] C_e \sqrt{\frac{\sigma_d - \sigma_s}{\sigma_s}} \left\{ 1 - \cos \left( \frac{\pi \sigma_d}{2 \sigma_s} \right) \right\} \left[ \frac{\cos^2 \theta + \left( \frac{a}{b} \right)^2 \sin^2 \theta}{1 + 1.464 \left( \frac{a}{b} \right)^{1.65}} \right]
\]

and \( L \) being given by twice the term \( r \) from Eq. (5) in Eq. (12a). The dimensionless fracture shear strain (Eq. (12b)) increases with driving stress, end zone yield strength, and Poisson’s ratio, and decreases with modulus and fracture aspect ratio. Values of fracture strain also depend on the position around the fracture tipline; for strain in the horizontal plane, \( \theta = 0^\circ \).

We note that the values of fracture shear strain \( \gamma \) in the 2-D approach also depend on the end-zone length, the maximum displacement, and, in turn, on the effective driving stress on the elastic–plastic fracture (Eq. (9)). In their study of faults, Cowie and Scholz (1992b) inferred driving stresses in the order of 100 MPa and a shear modulus of 10 GPa (resulting in \( G/\sigma_d = 100 \)). Smaller driving stresses (or ‘stress drops,’ e.g. <1 MPa; Martel and Pollard, 1989) may be used to represent individual seismic slip events having \( D_{\text{max}}/L = 10^{-3} – 10^{-5} \), whereas the larger cumulative geologic offsets on faults (Scholz, 1997) having \( D_{\text{max}}/L = 10^{-1} – 10^{-3} \) may be represented by cumulative driving stresses on the order of 100 MPa (e.g. Cowie and Scholz, 1992b). Using Eq. (11), the factor \( C \) required in the expression for \( \gamma \) (e.g. Scholz, 1997) is incorporated either by adjusting the magnitudes of the driving and yield stresses \( \sigma_d \) above 1 MPa (e.g. \( \sigma_d = 100 \) MPa and \( \sigma_s = (\sigma_d/\sigma_a) \times \sigma_d = (1.5 \times 100) = 150 \) MPa) or by reducing the modulus by an equivalent amount (e.g. \( G/100 = 16 \) MPa). The factor \( C_e \) and Eqs. (12a) and (12b) allows more precise values of fracture strain to be obtained from the \( D–L \) scaling relations.

Constant values of \( \gamma \) (and constant 45° slopes on the \( D–L \) diagram) are obtained for 3-D elliptical fractures that are tall compared with their lengths (2\( b \gg L \); see Fig. 13, dashed diagonal lines, left-hand side of diagram). The constant slopes imply that the tall 3-D fractures approximate 2-D ones that scale with length only, independently of their heights. The values of \( \gamma \) are influenced by the strength of the fracture’s end zone (Eqs. (12a) and (12b)). For given values of driving stress and rock-crack properties (module, Poisson’s ratio appropriate for sandstone; see caption to Fig. 12 for values), a weak end zone (\( \sigma_d/\sigma_s = 1.5 \)) leads to \( D_{\text{max}}/L = \alpha = 0.0017, \) or 0.17\% (bold curve in Fig. 13). \( D–L \) ratios increase for stronger end zones by a factor of \( \sim 4 \) at \( \sigma_d/\sigma_s = 50 \), appropriate to an LEFM fracture in perfectly elastic surroundings (Fig. 8). For end zone strengths between 1.5 and 2.5 (see Fig. 8), the \( D–L \) relation varies by ±50%.

The value of fracture shear strain (Eqs. (12a) and (12b)) accommodated by a fracture population described by the 3-D scaling relation remains constant for a given value of end-zone strength (other parameters held constant). However a constant fracture shear strain is not achieved for stratigraphic fractures (\( a/b \neq \text{constant} \)) because the slope of the \( D–L \) relation is nonlinear. This result parallels the decrease in a fracture’s ability to accommodate displacements as its aspect ratio increases (Willemse et al., 1996; Fig. 10) and implies that, in general, a fracture’s shape (\( a, b \)) must balance its maximum displacement (\( D_{\text{max}} \)) for a constant fracture strain to be maintained within a population.

In summary, for fractures that are much larger in height than in length (\( a \ll b \); left side of Fig. 14), the shorter (length) dimension controls the magnitude of displacement on the ‘tall’ fracture (e.g. Gudmundsson, 2000). In this case, \( D_{\text{max}} \) and \( L \) scale linearly, and 2-D scaling relations and

![Fig. 14. Comparison between 2-D and 3-D scaling relations. The 3-D results converge toward the 2-D values of \( \gamma \) (fracture shear strain, dashed lines in shaded region) for tall fractures (\( a/b \ll 0.2 \)). Curve shown calculated using \( D_{\text{max}}/L = 0.00275, b = 10 \) m, \( \sigma_d/\sigma_s = 2.0, \) \( \theta = 0^\circ, \) Poisson’s ratio of 0.4, and shear modulus of 1.6 GPa (\( G/\sigma_s = 80.3 \)).](image)
analyses such as those in use in the literature apply (e.g. Cowie and Scholz, 1992b; Bürgmann et al., 1994; Cladouhos and Marrett, 1996; Clark and Cox, 1996; Schlische et al., 1996). For long fractures \((a \gg b\); right side of Fig. 14), the smaller fracture height dominates the control on displacement, leading to a shallower, nonlinear slope on the \(D_{\text{max}}/L\) diagram (Fig. 14). Three-dimensional scaling relations apply where length is more than \(\approx 5\) times the height (i.e. \(a/b > 0.2\); unshaded region of Fig. 14).

5. Application to deformation bands in Utah

The deformation bands that are confined to layer \(e_1\) of the Entrada Sandstone define a characteristically shallow slope (<1) on the \(D-L\) diagram (Fossen and Hesthammer, 1997) over the full range of band lengths. This data set, which motivated the 3-D analysis of \(D-L\) scaling, allows the predictions of population strain and displacement-length scaling of elliptical fractures to be tested against field data.

In order to independently evaluate the predictions of fracture strain from the 3-D scaling relationship (Eq. (12b)), we measured the frequency and normal dip-slip offsets of a representative sample of unfaulted deformation bands from the outcrop (Figs. 2b and c and 3a). Fifty-eight bands were measured along a 50-m-long linear (and horizontal) traverse oriented at 220° (Fig. 15b and c), normal to the strike of the fault surface (that nucleated within a cluster of deformation bands) shown in Fig. 3e. The bands define two complementary (conjugate) sets of orientations striking \(\approx 280°\) and dipping 60–85° NE and SW (Fig. 15a), and plane subhorizontal extension along the direction perpendicular to the mean azimuth of the bands (200 ± 10°; star in Fig. 15a). Using the calculated horizontal components of offset obtained from the strike (e.g. Peacock and Sanderson, 1993; Priest, 1993, pp. 96–97) and dip of each band, we find a cumulative extensional (horizontal) strain of 0.12% (Fig. 15c) along 200° (star in Fig. 15c).

In order to compare the fracture shear strains from the population statistics to the measured strain along the traverse, the predicted band-parallel fracture shear strains \(\alpha\) (Eqs. (12a) and (12b)) are converted to the horizontal fracture strains \(\epsilon_h\) by using:

\[
\epsilon_h = \alpha(\cos \delta)
\]

in which \(\delta\) is the average dip of the bands and \(\theta = 0°\) (in Eq. (11)). Using the average dip angle of 65° measured along the traverse, the horizontal (extensional) component of the fracture strain \(\epsilon_h\) is reduced to 43% of the band-parallel fracture shear strain \(\alpha\). We varied the parameters in Eqs. (12a) and (12b) to determine the range of values consistent with the \(D-L\) scaling relations (Fossen and Hesthammer, 1997) and horizontal strain measured independently along the traverse. To achieve a horizontal fracture strain \(\epsilon_h\) of 0.12%, the values are \(3.0 < E < 5.0\) GPa (corresponding to a shear modulus \(G\) of 1.1–1.8 GPa), \(0.2 < \nu < 0.4\), \(1.5 < \sigma_f/\sigma_d < 2.5\), and \(60° < \delta < 70°\). The variations listed for Young’s modulus \(E\) (±25%), Poisson’s ratio (±50%), end zone strength \(\sigma_f/\sigma_d\) (±25%), and band dip angle (±10%) collectively

![Fig. 15](image-url)
Fig. 16. $D_{\text{max}}/L$ data for deformation bands plotted as a function of fracture aspect ratio $a/b$, indicating dependence of $D_{\text{max}}/L$ scaling of deformation bands on fracture shape rather than just their lengths ($L = 2a$). Data from Fossen and Hesthammer (1997). Tick marks on symbols show direction of qualitative uncertainties in $D_{\text{max}}$ associated with linkage geometries.

lead to an uncertainty in $\varepsilon_h$ of less than a factor of two. The horizontal fracture strain can be increased by using shallower dips, smaller modulus, smaller Poisson’s ratio, and/or stronger end zones. Increasing the modulus by a factor of 10 (keeping band dip and Poisson’s ratio constant, as above) also predicts the strain value (0.12%) as long as the driving stress is also increased by a factor of 10, so that $G/\sigma_3 = 80.3$ in both cases. However, increasing the end-zone strength to LEFM levels ($\sigma_3/\sigma_d > 50$) is not sufficient by itself to balance the larger value of $G$ and thereby predict the same value of fracture strain (0.12%). Given particular values of rock properties ($G$, $v$) and end zone strength ($\sigma_3/\sigma_d$), the driving stress $\sigma_3$ must be varied to predict particular values of fracture strain using Eq. (11).

The 73 measurements of $D_{\text{max}}$ and $L$ reported by Fossen and Hesthammer (1997) from the 9-m-thick layer within the Entrada Sandstone (“e1” in Figs. 2c and 3) are plotted as a function of aspect ratio in Fig. 16 (using a maximum thickness $2b$ of the porous sandstone layer of 9 m). Values of $D_{\text{max}}/L$ that are independent of aspect ratio, and hence the vertical fracture height, would cluster about horizontal lines on the diagram; such clustering is not observed. In contrast, the data suggest a reduction in $D_{\text{max}}/L$ with increasing values of fracture aspect ratio $a/b$ by a factor of at least 5–10, particularly for aspect ratios greater than one. We infer that the 3-D fracture geometry influences $D_{\text{max}}/L$ ratios within the study area.

Deformation bands coded by their map-view geometry (on the subhorizontal surface of the outcrop (Fig. 3a); Fossen and Hesthammer, 1997) are shown in Fig. 16. Isolated deformation bands (filled diamonds in Fig. 16) are widely separated from other bands. The lengths of closely spaced bands that interact mechanically with others (called ‘soft-linked’ in the literature) to form stepovers and eye structures (Cruikshank et al., 1991; Antonellini and Aydin, 1995; Fossen and Hesthammer, 1997) are difficult to assess precisely without quantifying the degree of interaction for each band segment (e.g. Segall and Pollard, 1980). Band segments in a soft-linked array plotted by using their individual lengths (open triangles in Fig. 16) exhibit excessive values of $D_{\text{max}}$ for their lengths (tick marks adjoining symbols in Fig. 16) because the influence of the other segments is not included (e.g. Segall and Pollard, 1980; Willemse, 1997). On the other hand, the aggregate length of the interacting array (open circles in Fig. 16) will appear underdisplaced relative to a continuous band of equivalent length. Linked arrays of deformation band segments (‘hard-linked systems’; filled squares in Fig. 16) may also exhibit somewhat smaller values of $D_{\text{max}}$ than found for continuous bands with the reduction related to the degree of roughness and linkage history of the band. Additional uncertainties could be added to all four groups of deformation bands to account for linkage geometries in the vertical, down-dip direction given appropriate observations (not available from the study area).

The data replotted on a traditional displacement–length diagram (Fig. 17) show that a power-law slope of approximately 0.5 (heavy line in Fig. 17a) could be used to describe the population (Fossen and Hesthammer, 1997, 1998). This best-fit slope of 0.5 (Fossen and Hesthammer, 1997) is significantly shallower than power-law slopes of 1.0 (dashed line in Fig. 17a) inferred for faults from a wide range of rock types (Cowie and Scholz, 1992a; Clark and Cox, 1996). We infer that the shallower slope occurs because the deformation bands—as a population—do not maintain a constant aspect ratio over the length scale shown.

According to the calculations reported by Willemse et al. (1996) and others, fractures having heights less than 3–5 times their lengths should scale as 3-D surfaces. Inspection of Fig. 17a shows that bands having $L < \sim 3$ m, or 1/3 the stratigraphic thickness, may be qualitatively consistent with the 2-D proportional growth slope of 1.0 (dashed line in Fig. 17a). Proportional growth of penny-shaped fractures ($a = b$) would follow a slope of 1.0, as demonstrated in Fig. 13b, by cutting across curves for different values of $b$. Because the slopes of the proportional and nonproportional curves are parallel for aspect ratios less than one, however, it may not be possible to distinguish between them for this range of relative lengths using statistics alone (see also Fig. 14). On the other hand, the scaling of longer deformation bands is consistent with nonproportional growth and constant fracture height. Given these uncertainties, we conclude that nonproportional growth will be most clearly demonstrated on a $D_{\text{max}}/L$ diagram for fractures having aspect ratios greater than one (Fig. 17b); when
combined with shorter fractures of the same population, the aggregate power-law slope for the data set will be less than one.

The value of horizontal fracture strain obtained from the traverse (0.12%) agrees with the range predicted for the deformation band population from the 3-D analysis (Fig. 17b) using the parameters noted above. Given this correspondence, the yield strength of the end zones bounding deformation bands in the study area is probably small ($\sigma_y / \sigma_d < 2.5$; see Figs. 8 and 13). Significantly larger values of end-zone strength would be associated with steeper near-tip displacement gradients (see Fossen and Hesthammer, 1997, for measurements) and with pervasive secondary structures nucleating near the tips (e.g. Cooke, 1997) of individual, isolated bands; neither of these is observed in the study area. A ratio of shear modulus to driving stress ($G / \sigma_d$) of 80–800 is implied by the values of fracture strain, end-zone strength, and linkage geometry (Figs. 16 and 17) for the population of deformation bands.

Fig. 17. Displacement–length scaling of deformation bands (data from Fossen and Hesthammer, 1997); symbols and tick marks as in Fig. 16. (a) Linear fits to the data; proportional growth curve (dashed line, slope = 1.0), least-squares regression (gray curve), $D_{\text{max}} = 0.002 L^{1.16}$, $r^2 = 0.775$. (b) 3-D scaling relations fitted to the data calculated using Eq. (11) with $2b = 9$ m, $\sigma_y / \sigma_d = 2.0$, $\theta = 0^\circ$. Poisson’s ratio of 0.4, $\delta = 65^\circ$ and shear modulus of 1.6 GPa ($G / \sigma_d = 80.3$). Gray curve shows value of horizontal fracture strain from traverse (0.12%).

Fig. 18. Examples of fractures growing at constant aspect ratio (proportional growth, a–d), for which traditional $D_{\text{max}} / L$ scaling relationships and constant slopes of 1.0 may apply, and fractures growing with increasing aspect ratio (e) and (f). Ellipses represent individual fracture surfaces, rectangles are interacting and/or linked fracture segments.
6. Discussion and implications

In order for fractures to define a linear scaling relation between maximum displacement and map length, and thereby to define a slope of one on log–log displacement–length diagrams, they must grow in size either two-dimensionally— as ‘tall’ fractures—or proportionally, either of which implies growth at a constant aspect ratio (Fig. 18a–d). Proportional growth requires that 3-D (elliptical) fractures increase in height as they lengthen. The range of variations in a fracture’s aspect ratio contributes a factor of perhaps 5–10 in its displacement values, potentially leading to this degree of scatter on $D_{max}/L$ diagrams (e.g., Willems et al., 1996; Guðmundsson, 2000). The scatter should be systematic, however, given that increasing fracture aspect ratio is associated with decreasing displacement magnitudes per unit fracture length. This effect is consistent both qualitatively and quantitatively with the shallower slope of $D_{max}/L$ data for deformation bands that are confined within a particular layer ($e_1$) of the Entrada Sandstone (Fossen and Hesthammer, 1997; Fig. 17).

Confinement of fractures such as deformation bands (e.g., Aydin, 1978b; Davis, 1999), small faults (e.g., Gross, 1995; Gross et al., 1997; Ackermann et al., 2001; Wilkins and Gross, 2002), and joints (e.g., Bahat and Engelder, 1984; Engelder, 1987; Pollard and Aydin, 1988; Helgeson and Aydin, 1991) to discrete stratigraphic layers (e.g., Fig. 4c) or by crustal rheology (e.g., Scholz, 1982; Cowie, 1998) has been well documented in the field and in the literature. This confinement, and associated preferential growth to larger aspect ratios, can occur when the stress and/or lithologic conditions in adjacent stratigraphic layers are unfavorable for propagation of the fracture into those layers (Fig. 18e and f). Assuming a uniform driving stress, the propagation energy (Pollard and Aydin, 1988) along elliptical fractures is greatest along their (smaller) semi-minor axis directions (Irwin, 1962; Kassir and Sih, 1966), leading to steeper displacement gradients and longer end zones in that direction (2b) than elsewhere along the periphery. However, propagation of fractures that impinge on stratigraphic contacts between dissimilar layers will be enhanced or impeded depending on the relative material properties and associated stress states (e.g., Pollard and Aydin, 1988; Helgeson and Aydin, 1991). Impeded fault propagation can lead to increased values of displacement, steeper
displacement gradients (Gupta and Scholz, 2000), and, if the stratigraphic thickness along this (2b) direction is smaller than that required for the end zone to balance the near-tip stresses, smaller end zones (e.g. Martel and Pollard, 1989). The fracture will be trapped within its layer if these elevated near-tip stresses cannot meet the new yield criteria across the interface (e.g. Cooke and Underwood, 2001), leading to nonproportional growth.

Initially, growth of bands that are sufficiently small to not impinge on layer boundaries may be proportional (Fig. 19a). Trapping of the band within a layer leads to increasing aspect ratio, nonproportional growth, and a shallowing slope on the $D_{\text{max}}/L$ diagram (Fig. 19a; Fossen and Hesthammer, 1997, 1998). Once a faulted band breaks through the layer, however, it can follow a steeper trajectory on the $D_{\text{max}}/L$ diagram until it becomes confined within a thicker sequence (Fig. 19a). Fossen and Hesthammer (1997) inferred a steeper growth path (e.g. Nicol et al., 1996) that connected deformation bands within the Entrada Sandstone to the larger faulted bands studied by Krantz (1988). We suggest that a decrease in aspect ratio, leading to the steeper growth path (Fig. 19a), was enabled by the nucleation of fault slip surfaces on the bands that facilitated their ability to cut across stratigraphic boundaries.

Linkage of deformation bands, or fractures in general, leads to trajectories on the $D_{\text{max}}/L$ diagram analogous to the effect of a confining stratigraphy (Fig. 19b). Because along-strike (horizontal) linkage serves to increase aspect ratio, linked arrays behave as ‘longer’ elliptical fractures that can accommodate less displacement per unit map length than their 2-D (‘tall’) counterparts, as recently demonstrated in numerical simulations (Willemsen and Pollard, 2000). Fractures and arrays that increase in length $L$ by segment linkage must follow a progressively shallowing growth path on the $D_{\text{max}}/L$ diagram (Fig. 19b). On the other hand, down-dip (vertical) linkage increases $2b$ instead of $2a$, leading to a decrease in aspect ratio (to ‘taller’ fractures) and an increase in the relative amount of displacement accommodated along the aggregate fracture. Down-dip linkage (by itself, with no increase in $L$) would be associated with vertical growth paths on the $D_{\text{max}}/L$ diagram (Fig. 19b). These growth paths, defined by the aggregate fracture aspect ratio, provide a physical basis for the ‘stair-step’ growth sequence proposed by Cartwright et al. (1995). Fracture growth by segment linkage in three dimensions would follow a unit slope if the rates of segment linkage were comparable, on average, in both directions (shaded bar in Fig. 19). Preferential linkage in either direction would lead to displacement discrepancies relative to 2-D expectations and consequent scatter on the $D_{\text{max}}/L$ diagram.

7. Conclusions

The magnitude of maximum displacement along a fracture depends on both dimensions of its geometry (length and height) in addition to driving stress and rock mass properties. As a result, relating $D_{\text{max}}$ to map length (or down-dip height) alone provides an incomplete assessment of the control of displacement magnitude by this fracture dimension. Although $D_{\text{max}}/L$ diagrams relate only two of the three key parameters ($D_{\text{max}}$ and $L$ but not $H$), the dependence of $D_{\text{max}}$ on aspect ratio can be represented, with a small increase in complexity, on these diagrams (Figs. 13 and 14). Because particular mechanisms such as mechanical interaction and linkage affect the aggregate shapes of fractures and fracture arrays in 3-D, scatter on $D_{\text{max}}/L$ diagrams can represent variations in fracture aspect ratio.

The deformation bands investigated in this paper are confined to a particular layer within the Entrada Sandstone, and hence have a fixed maximum height. Increases in their map lengths require that their aspect ratios also increase, leading to a systematic reduction in the amount of displacement they can accommodate per unit length and shallower slopes on the $D_{\text{max}}/L$ diagram. Formation and growth of deformation bands is sensitive to rock properties, similar to other classes of fractures such as anticracks (compaction seams, stylolites) and dilatant cracks. A nonlinear 3-D displacement–length scaling relation is likely to be associated also with these classes of structures, rather than a linear one on a $D_{\text{max}}/L$ diagram, provided their heights remain limited by the layer thickness. Because faults (with well-developed slip surfaces) readily cut across bedding, scaling relations for faults (or faulted deformation bands) are less strongly affected by stratigraphy, leading to fault shapes that produce a unit power-law slope on the $D_{\text{max}}/L$ diagram. As a result, standard displacement–length scaling relations (with unit slopes) are most applicable to faults.

The hypotheses presented in this paper can be tested by measuring $D_{\text{max}}$, $a$, and $b$ for deformation bands, faulted bands, and other fracture types in relation to the layer thickness. Stratigraphically or rheologically confined fractures that are sufficiently long for their aspect ratios to exceed 5–10 should provide reliable data sets for a 3-D analysis of displacement–length scaling. We anticipate that other fracture types, such as dilatant cracks and stylolites (or compaction seams), should exhibit aspect-ratio scaling if they are mechanically confined to particular beds, leading to progressively larger aspect ratios and, potentially, nonproportional displacement–length scaling.

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